An Analytical Approach for Structural Synthesis of Substructures

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A structure is broken down into a number of substructures by means of the finite element method and the substructures are synthesized for the complete structure. The divided substructures take two types : fixed-free and free-free elements. The flexibility and stiffness matrices of the free-free elements are the Moore-Penrose inverse of each other. Thus, it is not easy to determine the equilibrium equations of the complete structure composed of two mixed types of substructures. This study provides the general form of equilibrium equation of the entire structure through the process of assembling the equilibrium equations of substructures with end conditions of mixed types. Applications demonstrate that the proposed method is effective in the structural analysis of geometrically complicated structures.

Key Words : Substructuring, Compatibility, Equilibrium Equation, Constraint, Synthesis Method

1. Introduction

Finite element models are utilized to obtain solutions for the displacements of structures subjected to static loads. Complete structures are very complex, and the finite element model of the entire structure might contain so many degrees of freedom that it would be infeasible to perform a structural analysis based on the finite element equations for the complete system of disturbed stress state and geometrical irregularity. Major components are often designed and produced by different organizations, and a finite element model of the entire structure is assembled.

The static analysis of complex structures often needs to be performed by means of substructuring. Substructuring identifies the process of the subdivision of the overall structure into two or more substructures. Greater emphasis has been placed on the study of modeling the static behavior of individual substructures.

The substructure synthesis method requires two steps : (1) the determination of the equilibrium equation of each substructure ; (2) the substructure synthesis to consider compatibility conditions at the interfaces. The end conditions of substructures can be considered as two types : fixed-free and free-free elements. The free-free elements represent the rigid body modes. Although the synthesis process of each substructure requires the calculation of the displacement vector, the displacements of each free-free element can not be explicitly obtained because the flexibility and stiffness matrices are the Moore-Penrose inverse of each other.

The substructure synthesis approach including free-free substructures depends on numerical

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approaches to use Lagrange multipliers. There have been a few methods to develop approximate mathematical models for assembling the static behavior of each substructure. Based on a decomposition of the finite element model into substructures, Felippa and Park (1997) presented a direct flexibility method. Park, Justino F, and Felippa (1997) introduced an algebraically patitioned FETI method for the solution of structural engineering problems. They (1998) also provided flexibility expressions for symmetric and unsymmetric free-free stiffnesses. However, this approach has a difficulty in determining the explicit form of Moore-Penrose relation of flexibility and stiffness matrices. Farhat, Lacour, and Rixen (1998) gave the iterative solution by substructuring methods of the large-scale systems of equations of the substructures discretized by linear multipoint constraints.

The aim of this study is to derive the equilibrium equation of an entire structure subjected to linear constraints by extremizing the potential energy to consider constraint equations and utilizing fundamental linear algebra. Modifying the derived equation, this study presents the general form of the equilibrium equation of the entire structure composed of free-free and fixed-free substructures subjected to compatibility conditions or other types of linear constraints. Applications demonstrate that the proposed method is effective and simple.

2. Equilibrium Equations of Constrained Structures

The static equilibrium equation is derived by minimizing total potential energy with respect to displacement vector. To do this, let us consider an n degrees of freedom structure with the total potential energy expressed as

$$\Pi=\frac{1}{2}\mathbf{u}^{T}\mathbf{K}\mathbf{u}-\mathbf{u}^{T}\mathbf{F} \tag{1}$$

in which Π denotes the potential energy of the structure, $\mathbf{u}$ and $\mathbf{F}$ are the $n\times 1$ generalized displacement and nodal force vectors. $\mathbf{K}$ was assumed as the $n\times n$ positive definite stiffness matrix for this derivation. From equation (1), the equilibrium equation is obtained as

$$\mathbf{K}\mathbf{u}=\mathbf{F} \tag{2}$$

The existence of constraints like compatibility leads to new type of equilibrium equation because equation (1) and the constraints should be combined. Assume that the structure is subjected to m linear constraints written in matrix form as

$$\mathbf{A}\mathbf{u}=\mathbf{b}, \qquad m<n \tag{3}$$

where $\mathbf{A}$ is $m\times n$ real matrix and $\mathbf{b}$ is an $m\times 1$ vector. Although the equilibrium equations of the structure are given by equation (2), the real displacement responses of the structure must satisfy equation (3) by the existence of constraints. It indicates that the real responses should be determined by combining equations (1) and (3). To combine equations (1) and (3), they are rewritten as

$$\Pi=\frac{1}{2}(\mathbf{K}^{1/2}\mathbf{u})^{T}(\mathbf{K}^{1/2}\mathbf{u})-\mathbf{u}^{T}\mathbf{F} \tag{3a}$$

$$\mathbf{A}\mathbf{K}^{-1/2}\mathbf{K}^{1/2}\mathbf{u}=\mathbf{b} \tag{3b}$$

respectively. The general solution* of equation (3b) with respect to $\mathbf{K}^{1/2}\mathbf{u}$ is obtained as

$$\begin{aligned}\mathbf{K}^{1/2}\mathbf{u}=&(\mathbf{A}\mathbf{K}^{-1/2})^{+}\mathbf{b}\\&+\lfloor\mathbf{I}-(\mathbf{A}\mathbf{K}^{-2/2})^{+}(\mathbf{A}\mathbf{K}^{-1/2})\rfloor\,\mathbf{y}\end{aligned} \tag{4}$$

where '+' denotes the Moore-Penrose inverse, $\mathbf{y}$ is an arbitrary vector, and $\mathbf{I}$ is an identity matrix.

Substitution of equation (4) into equation (1a) leads to

$$\begin{aligned}\Pi=&\frac{1}{2}[(\mathbf{A}\mathbf{K}^{-1/2})^{+}\mathbf{b}+\{\mathbf{I}-(\mathbf{A}\mathbf{K}^{-1/2})^{+}(\mathbf{A}\mathbf{K}^{-1/2})\}\mathbf{y}]^{T}\\&[(\mathbf{A}\mathbf{K}^{-1/2})^{+}\mathbf{b}+\{\mathbf{I}-(\mathbf{A}\mathbf{K}^{-1/2})^{+}(\mathbf{A}\mathbf{K}^{-1/2})\}\mathbf{y}]\\&-[(\mathbf{A}\mathbf{K}^{-1/2})^{+}\mathbf{b}+\{\mathbf{I}-(\mathbf{A}\mathbf{K}^{-1/2})^{+}(\mathbf{A}\mathbf{K}^{-1/2})\}\mathbf{y}]^{T}\mathbf{K}^{-1/2}\mathbf{F}\end{aligned} \tag{5}$$

Minimizing equation (5) with respect to the arbitrary vector $\mathbf{y}$ and using $(\mathbf{A}\mathbf{K}^{-1/2})^{+}(\mathbf{A}\mathbf{K}^{-1/2})(\mathbf{A}\mathbf{K}^{-1/2})^{+}=(\mathbf{A}\mathbf{K}^{-1/2})^{+}$, the result can be written as

*The general solution of $\mathbf{A}\mathbf{x}=\mathbf{b}$, where $\mathbf{A}$ is $m\times n$ matrix, $\mathbf{x}$ and $\mathbf{b}$ are $n\times 1$ and $m\times 1$ vectors, respectively, can be written as

$$\mathbf{x}=\mathbf{A}^{+}\mathbf{b}+\lfloor\mathbf{I}+\mathbf{A}^{+}\mathbf{A}\rfloor\,\mathbf{d}$$

where $\mathbf{I}$ is $n\times n$ identity matrix and $\mathbf{d}$ is $n\times 1$ arbitrary vector.

$$\frac{\partial \Pi}{\partial \mathbf{y}}=[\mathbf{I}-(\mathrm{AK}^{-1/2})^{+}(\mathbf{AK}^{-1/2})]\mathbf{y} \\ -[\mathbf{I}-(\mathbf{AK}^{-1/2})^{+}(\mathbf{AK}^{-1/2})]\mathbf{K}^{-1/2}\mathbf{F}=0 \tag{6}$$

Solving equation (6) with respect to the arbitrary vector $\mathbf{y}$, and utilizing $\lfloor \mathbf{I}-(\mathbf{AK}^{-1/2})^{+}(\mathbf{AK}^{-1/2})\rfloor \lfloor \mathbf{I}-(\mathbf{AK}^{-1/2})^{+}(\mathbf{AK}^{-1/2})\rfloor=\lfloor \mathbf{I}=(\mathbf{AK}^{-1/2})^{+}(\mathbf{AK}^{-1/2})\rfloor$ and $[\mathbf{I}-(\mathbf{AK}^{-1/2})^{+}(\mathbf{AK}^{-1/2})]^{+}=[\mathbf{I}-(\mathbf{AK}^{-1/2})^{+}(\mathbf{AK}^{-1/2})]$ into the result, it can be calculated as

$$\mathbf{y}=\lfloor \mathbf{I}-(\mathbf{AK}^{-1/2})^{+}(\mathbf{AK}^{-1/2})\rfloor \mathbf{K}^{-1/2}\mathbf{F} \\ +(\mathbf{AK}^{-1/2})^{+}(\mathbf{AK}^{-1/2})\mathbf{z} \tag{7}$$

where $\mathbf{z}$ is another arbitrary vector. Substituting equation (7) into equation (4) and arranging the result with $(\mathbf{AK}^{-1/2})^{+}(\mathbf{AK}^{-1/2})(\mathbf{AK}^{-1/2})^{+}=(\mathbf{AK}^{-1/2})^{+}$, we obtain the static equilibrium equation to satisfy the constraints (3) as

$$\mathbf{u}=\mathbf{K}^{-1}\mathbf{F}+\mathbf{K}^{-1/2}(\mathbf{AK}^{-1/2})^{+}(\mathbf{b}-\mathbf{AK}^{-1}\mathbf{F}) \tag{8}$$

However, the derived equation (8) has a limitation not to be able to describe the static behavior of the substructures of rigid body modes that the stiffness matrix of equation (2) is positive semidefinite matrix with a rank deficiency of at most one.

Premultiplying $\mathbf{K}$ on both sides of equation (8), the result is derived as

$$\mathbf{Ku}=\mathbf{F}+\mathbf{F}^{c} \tag{9}$$

where the second term of the right-hand side defines the constraint force vector expressed as

$$\mathbf{F}^{c}=\mathbf{K}^{1/2}(\mathbf{AK}^{-1/2})^{+}(\mathbf{b}-\mathbf{AK}^{-1}\mathbf{F}) \tag{10}$$

The constraint force is the force required for satisfying the constraint like equation (3). In this study, the physical meaning of the constraint force was investigated. Substitution of equation (9) into equation (1) yields

$$\Pi=\frac{1}{2}[\mathbf{F}+\mathbf{F}^{c}]^{T}\mathbf{K}^{-1}[\mathbf{F}+\mathbf{F}^{c}] \\ -[\mathbf{F}+\mathbf{F}^{c}]^{T}\mathbf{K}^{-1}\mathbf{F} \tag{11}$$

Minimizing the potential energy with respect to the constraint force, it leads to

$$\frac{\partial \Pi}{\partial \mathbf{F}^{c}}=\mathbf{K}^{-1}[\mathbf{F}+\mathbf{F}^{c}]-\mathbf{K}^{-1}\mathbf{F}=\mathbf{K}^{-1}\mathbf{F}^{c}=\mathbf{0} \tag{12}$$

Premultiplying $\mathbf{K}^{1/2}$ on both sides of equation (4) and using equation (9) into equation (12), it follows that

$$\mathbf{F}^{c}=-\mathbf{F}+\mathbf{K}^{1/2}(\mathbf{AK}^{-1/2})^{+}\mathbf{b} \\ +\mathbf{K}^{1/2}\lfloor \mathbf{I}-(\mathbf{AK}^{-1/2})^{+}(\mathbf{AK}^{-1/2})\rfloor \mathbf{y} \tag{13}$$

Substituting equation (13) into equation (12), defining $\mathbf{Q}=\lfloor \mathbf{I}-(\mathbf{AK}^{-1/2})^{+}(\mathbf{AK}^{-1/2})\rfloor$ and $\mathbf{Q}^{+}\mathbf{Q}=\mathbf{Q}$, and solving the result with respect to the arbitrary vector $\mathbf{y}$, it is derived as

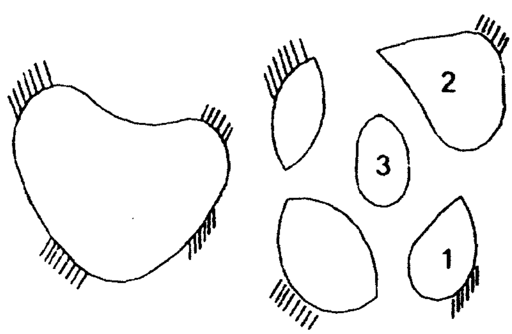
$$\mathbf{y}=\mathbf{Q}\lfloor \mathbf{K}^{-1/2}\mathbf{F}+(\mathbf{AK}^{-1/2})^{+}\mathbf{b}\rfloor \\ +(\mathbf{AK}^{-1/2})^{+}(\mathbf{AK}^{-1/2})\mathbf{z} \tag{14}$$

where $\mathbf{z}$ is another arbitrary vector. The final result to substitute equation (14) into equation (13) defines the constraint force vector written as equation (10).

From the above derivation, the constraint forces are defined as the minimum forces of all forces provided by nature to satisfy the constraints. Also, it is realized that the equilibrium equation of constrained structures and the constraint forces can be obtained by minimizing the total potential energy with respect to the displacement vector and the constraint force vector, respectively.

3. Generalized Method for Structural Synthesis of Substructures

Using the proposed equation, this section determines the equilibrium equation of a complete structure to be composed of various substructures. Figure 1 exhibits a complete structure and its partitioned substructures. The equilibrium equation of each substructures on n_s partitioned



(a) (b)

Fig. 1 A complete structure composed of n_s substructures; (a) A complete structure, (b) n_s substructures

substructures can be written in matrix form as

$$\mathbf{K}^s\mathbf{u}^s=\mathbf{f}^s,\ \mathbf{s}=1,\ 2,\ \cdots,\ n_s \tag{15}$$

where $\mathbf{K}^s=\begin{bmatrix}\mathbf{K}_{ii}^s & \mathbf{K}_{ib}^s\\ \mathbf{K}_{ib}^s & \mathbf{K}_{bb}^s\end{bmatrix}$, $\mathbf{u}^s=\begin{bmatrix}\mathbf{u}_i^s\\ \mathbf{u}_b^s\end{bmatrix}$, $\mathbf{f}^s=\begin{bmatrix}\mathbf{f}_i^s\\ \mathbf{f}_b^s\end{bmatrix}$ and the subscripts i and b represent the interior and boundary region of each substructures, respectively. And the vector $\mathbf{u}$ denotes the displacement vector.

The equilibrium equation of the complete structure is determined by applying compatibility conditions at the boundaries. The compatibility conditions are written as

$$\mathbf{B}^s\mathbf{u}^s=\mathbf{0} \tag{16}$$

where $\mathbf{B}^s$ is Boolean matrix. Substitution of equations (15) and (16) into equation (8) yields the equilibrium equation of the complete structure. It is observed that the constrained behavior of structures can be easily and simply determined by substituting into equation (8).

One of the substructure synthesis methods is static condensation method. In the following, the equilibrium equation by static condensation is derived and compared with the proposed method.

Consider a simple structure subdivided into two substructures as shown by Fig. 2. The equilibrium equation of the complete structure is written as

$$\begin{bmatrix}\mathbf{K}_{ii}^1 & \mathbf{K}_{ib}^1 & \mathbf{0}\\ \mathbf{K}_{bi}^1 & \mathbf{K}_{bb}^1+\mathbf{K}_{bb}^2 & \mathbf{K}_{bi}^2\\ \mathbf{0} & \mathbf{K}_{ib}^2 & \mathbf{K}_{ii}^2\end{bmatrix}\begin{bmatrix}\mathbf{u}_i^1\\ \mathbf{u}_b^1\\ \mathbf{u}_i^2\end{bmatrix}=\begin{bmatrix}\mathbf{f}_i^1\\ \mathbf{f}_b^1\\ \mathbf{f}_i^2\end{bmatrix} \tag{17}$$

The equilibrium equation of each substructures can be expressed as

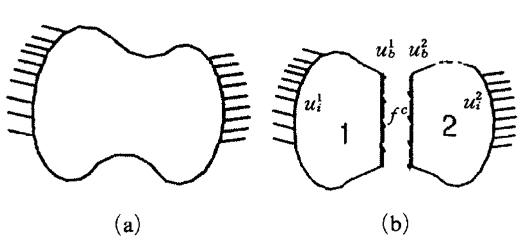


Fig. 2 A simple structure divided into two substructures ; (a) A simple structure, (b) Partitioned substructures

$$\begin{bmatrix}\mathbf{K}_{ii}^1 & \mathbf{K}_{ib}^1\\ \mathbf{K}_{bi}^1 & \mathbf{K}_{bb}^1\end{bmatrix}\begin{bmatrix}\mathbf{u}_i^1\\ \mathbf{u}_b^1\end{bmatrix}=\begin{bmatrix}\mathbf{f}_i^1\\ \mathbf{f}_b^1+\mathbf{f}^c\end{bmatrix} \tag{18a}$$

$$\begin{bmatrix}\mathbf{K}_{bb}^2 & \mathbf{K}_{bi}^2\\ \mathbf{K}_{ib}^2 & \mathbf{K}_{ii}^2\end{bmatrix}\begin{bmatrix}\mathbf{u}_b^2\\ \mathbf{u}_i^2\end{bmatrix}=\begin{bmatrix}-\mathbf{f}^c\\ \mathbf{f}_i^2\end{bmatrix} \tag{18b}$$

where $\mathbf{f}^c$ is the constraint force vector required to satisfy compatibility at the boundary, and the superscripts 1 and 2 indicate the substructure. Determining the constraint force vector from equation (18b) and substituting the result into equation (18a), the equilibrium equation of the entire substructure is obtained by the static condensation. Solving the second equation of equation (18b) with respect to $\mathbf{u}_i^2$, it is derived as

$$\mathbf{u}_i^2=(\mathbf{K}_{ii}^2)^{-1}(\mathbf{f}_i^2-\mathbf{K}_{ib}^2\mathbf{u}_b^2) \tag{19}$$

Inserting equation (19) into the first equation of equation (18b), the constraint force vector is derived in terms of

$$\mathbf{f}^c=\lfloor\mathbf{K}_{bb}^2-\mathbf{K}_{bi}^2(\mathbf{K}_{ii}^2)^{-1}\mathbf{K}_{ib}^2\rfloor\ \mathbf{u}_b^2+\mathbf{K}_{bi}^2(\mathbf{K}_{ii}^2)^{-1}\mathbf{f}_i^2 \tag{20}$$

Utilizing the equilibrium condition of the constraint forces at boundaries and substituting equation (20) into equation (18a), the equilibrium equation of the complete structure is obtained as

$$\begin{bmatrix}\mathbf{K}_{ii}^1 & \mathbf{K}_{ib}^1 & \mathbf{0}\\ \mathbf{K}_{bi}^1 & \mathbf{K}_{bb}^1 & -\mathbf{K}_{bb}^2+\mathbf{K}_{bi}^2(\mathbf{K}_{ii}^2)^{-1}\mathbf{K}_{ib}^2\\ \mathbf{0} & \mathbf{K}_{ib}^2 & \mathbf{K}_{ii}^2\end{bmatrix}\begin{bmatrix}\mathbf{u}_i^1\\ \mathbf{u}_b^1\\ \mathbf{u}_i^2\end{bmatrix}=\begin{bmatrix}\mathbf{f}_i^1\\ \mathbf{f}_b^1+\mathbf{K}_{bi}^2(\mathbf{K}_{ii}^2)^{-1}\mathbf{f}_i^2\\ \mathbf{f}_i^2\end{bmatrix} \tag{21}$$

Although equation (21) takes another type of the equilibrium equation of the complete structure, the numerical results coincide with equation (17).

If the rank of stiffness matrix $\mathbf{K}$ in equation (2) is less than n, the displacement responses cannot explicitly be obtained because there are infinite numbers of the inverse matrix of $\mathbf{K}$. In this study, the proposed equation is extended to the application of rigid body substructures.

Assume that the stiffness matrix $\mathbf{K}$ of equation (2) is positive semidefinite matrix. Because the inverse matrix of the stiffness matrix cannot be obtained, equation (2) is modified. Splitting the stiffness matrix into the diagonal matrix $\mathbf{K}_d$ and off-diagonal matrix $\mathbf{K}_o$, and arranging the result, it can be written as

$$\mathbf{K}_d\mathbf{u}=\mathbf{F}-\mathbf{K}_o\mathbf{u} \tag{22}$$

where $\mathbf{K}_d$ is full rank matrix. Assuming that the structure is constrained by the conditions like equation (3), and utilizing the equation (8), the equilibrium equation of constrained structure can be obtained by

$$\begin{aligned}\mathbf{u}=&\mathbf{K}_d^{-1}(\mathbf{F}-\mathbf{K}_o\mathbf{u})\\&+\mathbf{K}_d^{-1/2}(\mathbf{A}\mathbf{K}_d^{-1/2})^{+}[\mathbf{b}-\mathbf{A}\mathbf{K}_d^{-1}(\mathbf{F}-\mathbf{K}_o\mathbf{u})]\end{aligned} \tag{23}$$

Arranging equation (23) with respect to the displacement vector $\mathbf{u}$, the ultimate equilibrium equation yields

$$\mathbf{K}^*\mathbf{u}=\mathbf{F}^* \tag{24}$$

where

$$\mathbf{K}^*=\mathbf{I}+\lfloor\mathbf{I}-\mathbf{K}_d^{-1/2}(\mathbf{A}\mathbf{K}_d^{-1/2})^{+}\mathbf{A}\rfloor\,\mathbf{K}_d^{-1}\mathbf{K}_o \tag{25a}$$

$$\mathbf{F}^*=\mathbf{K}_d^{-1}\mathbf{F}+\mathbf{K}_d^{-1/2}(\mathbf{A}\mathbf{K}_d^{-1/2})^{+}(\mathbf{b}-\mathbf{A}\mathbf{K}_d^{-1}\mathbf{F}) \tag{25b}$$

Equation (24) represents the equilibrium equation of constrained structures including rigid body modes.

The static condensation method must be one of the analytical methods to reduce the degree of freedom and to synthesize several substructures into an entire structure. The proposed method has advantages to be able to easily establish the equilibrium equations by simultaneously substituting the equilibrium equations of all substructures and constraint equations into the governing equation without calculating the constraint forces in the formulation process.

In the following, the validity of equation (24) is illustrated.

4. Application I

Consider a four-spring system described by the displacement vector $\mathbf{u}=[u_1\ u_2\ u_3\ u_4]^T$ as shown in Fig. 3. The equilibrium equation of this system can be written by

$$\begin{bmatrix}k_1+k_2 & -k_2 & 0 & 0\\ -k_2 & k_2+k_3 & -k_3 & \beta\\ 0 & -k_3 & k_3+k_4 & -k_4\\ 0 & 0 & -k_4 & k_4\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix}=\begin{bmatrix}f_1\\f_2\\f_3\\f_4\end{bmatrix} \tag{26}$$

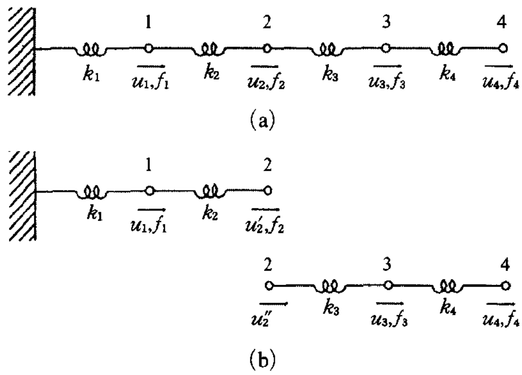


Fig. 3 A four-spring system ; (a) A complete spring system, (b) Two spring systems partitioned at node 2

where k_i and $f_i (i=1, 2, 3, 4)$ represent the stiffnesses of spring and the external forces, respectively. Assume that the system is subjected to a constraint written by

$$u_1+u_2+u_3+u_4=0 \tag{27}$$

The structural system constrained by equation (27) can be explicitly described by equation (8). For numerical results, the mechanical properties of the system were assumed as

$$\begin{aligned}&k_1=300,\ k_2=500,\ k_3=600,\ k_4=900\\&f_1=30,\ f_2=20,\ f_3=60,\ f_4=-70\end{aligned} \tag{28}$$

Substituting equations (26) and (27) into equation (8), utilizing the numerical values of equation (28) and MATLAB program, the constrained displacement values are calculated as

$$\begin{aligned}&u_1=0.0521,\ u_2=0.0355\\&u_3=-0.015,\ u_4=-0.0861\end{aligned} \tag{29}$$

It can be observed that the final values satisfy the constraint equation (27). Also, the constraint forces are calculated by equation (10) and they are obtained as

$$f_1^c=f_2^c=f_3^c=f_4^c=-6.096 \tag{30}$$

For investigating the validity of the proposed synthesis method (24), the system was divided into two subsystems at node 2. The equilibrium equation of each subsystems can be expressed as

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2' \end{bmatrix}=\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{31a}$$

$$\begin{bmatrix} k_3 & -k_3 & 0 \\ -k_3 & k_3+k_4 & -k_4 \\ 0 & -k_4 & k_4 \end{bmatrix}\begin{bmatrix} u_2'' \\ u_3 \\ u_4 \end{bmatrix}=\begin{bmatrix} 0 \\ f_3 \\ f_4 \end{bmatrix} \tag{31b}$$

where u_2' and u_2'' indicate the displacements at node 2 of the two subsystems, respectively. The flexibility matrix of equation (31b) can not be explicitly determined because the stiffness matrix is not full rank. The equilibrium equation of the complete system can be written as

$$\begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 & 0 \\ -k_2 & k_2 & 0 & 0 & 0 \\ 0 & 0 & k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2' \\ u_2'' \\ u_3 \\ u_4 \end{bmatrix}=\begin{bmatrix} f_1 \\ f_2 \\ 0 \\ f_3 \\ f_4 \end{bmatrix} \tag{32}$$

Splitting the stiffness matrix into the diagonal matrix and off-diagonal matrix and considering the compatibility and constraint (27), the parametric matrices of equation (25) are obtained as

$$\mathbf{K}_d = diag[k_1+k_2 \; k_2 \; k_3 \; k_3+k_4 \; k_4]$$

$$\begin{bmatrix} 0 & -k_2 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_3 & 0 \\ 0 & 0 & -k_3 & 0 & -k_4 \\ 0 & 0 & 0 & -k_4 & 0 \end{bmatrix}, \mathbf{A}=\begin{bmatrix} 0 & 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix} \tag{33}$$

Substituting the numerical values into equation (24), the final displacements are calculated as

$$\begin{aligned} u_1&=0.0521, \; u_2'=u_2''=0.0355 \\ u_3&=-0.015, \; u_4=-0.0861 \end{aligned} \tag{34}$$

The results coincide with the previous results (29). However, the proposed method shows that each displacement of two subsystems at node 2 was repeatedly calculated. Although this will cause the repeated calculation, the proposed method gives the explicit solutions without depending on any numerical analysis. Also, the time-consuming difficulty will be overcome by reducing the number of degrees of freedom of interior substructures through the static condensation approach.

5. Reduction of Degrees of Freedom by Static Condensation

The equilibrium equations of two substructures as shown by Fig. 2 are written as

$$\begin{bmatrix} \mathbf{K}^1_{ii} & \mathbf{K}^1_{ib} \\ \mathbf{K}^1_{bi} & \mathbf{K}^1_{bb} \end{bmatrix}\begin{bmatrix} \mathbf{u}^1_i \\ \mathbf{u}^1_b \end{bmatrix}=\begin{bmatrix} \mathbf{f}^1_i \\ \mathbf{f}^1_b \end{bmatrix} \tag{35a}$$

$$\begin{bmatrix} \mathbf{K}^2_{bb} & \mathbf{K}^2_{bi} \\ \mathbf{K}^2_{ib} & \mathbf{K}^2_{ii} \end{bmatrix}\begin{bmatrix} \mathbf{u}^2_b \\ \mathbf{u}^2_i \end{bmatrix}=\begin{bmatrix} 0 \\ \mathbf{f}^2_i \end{bmatrix} \tag{35b}$$

Assume that the substructure 2 exhibits free body mode. Decomposing the interior region of the substructure 1 into master and slave components $\mathbf{u}^s_{im}$ and $\mathbf{u}^s_{is}$, respectively, equation (35) is rewritten as

$$\begin{bmatrix} \mathbf{k}^1_{imim} & \mathbf{k}^1_{imis} & \mathbf{k}^1_{imb} & 0 & 0 \\ \mathbf{k}^1_{isim} & \mathbf{k}^1_{isis} & \mathbf{k}^1_{isb} & 0 & 0 \\ \mathbf{k}^1_{bim} & \mathbf{k}^1_{bis} & \mathbf{k}^1_{bb} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{k}^2_{bb} & \mathbf{k}^2_{bi} \\ 0 & 0 & 0 & \mathbf{k}^2_{ib} & \mathbf{k}^2_{bb} \end{bmatrix}\begin{bmatrix} \mathbf{u}^1_{im} \\ \mathbf{u}^1_{is} \\ \mathbf{u}^1_b \\ \mathbf{u}^2_b \\ \mathbf{u}^2_i \end{bmatrix}=\begin{bmatrix} \mathbf{f}^1_{im} \\ \mathbf{f}^1_{is} \\ \mathbf{f}^1_b \\ 0 \\ \mathbf{f}^2_i \end{bmatrix} \tag{36}$$

where the subscripts m and s denote master and slave modes, respectively.

In order to eliminate the slave modes of the substructure 1 and all modes of the substructure 2, equation (36) is partitioned as

$$[\mathbf{k}^1_{imim} \; \mathbf{k}^1_{imis} \; \mathbf{k}^1_{imb} \; 0 \; 0]\begin{bmatrix} \mathbf{u}^1_{im} \\ \mathbf{u}^1_{is} \\ \mathbf{u}^1_b \\ \mathbf{u}^2_b \\ \mathbf{u}^2_i \end{bmatrix}=[\mathbf{f}^1_{im}] \tag{37a}$$

$$\begin{bmatrix} \mathbf{k}^1_{isim} & \mathbf{k}^1_{isis} & \mathbf{k}^1_{isb} & 0 & 0 \\ \mathbf{k}^1_{bim} & \mathbf{k}^1_{bis} & \mathbf{k}^1_{bb} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{k}^2_{bb} & \mathbf{k}^2_{bi} \\ 0 & 0 & 0 & \mathbf{k}^2_{ib} & \mathbf{k}^2_{ii} \end{bmatrix}\begin{bmatrix} \mathbf{u}^1_{im} \\ \mathbf{u}^1_{is} \\ \mathbf{u}^1_b \\ \mathbf{u}^2_b \\ \mathbf{u}^2_i \end{bmatrix}=\begin{bmatrix} \mathbf{f}^1_{is} \\ \mathbf{f}^1_b \\ 0 \\ \mathbf{f}^2_i \end{bmatrix} \tag{37b}$$

Arranging equation (37b) with respect to the displacement vector $\mathbf{u}_c=[\mathbf{u}^1_{is} \; \mathbf{u}^1_b \; \mathbf{u}^2_b \; \mathbf{u}^2_i]$, and applying the results and compatibility conditions to equation (24), the coefficient matrices of equilibrium equation $\mathbf{K}^*\mathbf{u}_c=\mathbf{F}^*$ are derived as

$$\mathbf{K}^*=\mathbf{I}+\lfloor \mathbf{I}-\mathbf{K}_d^{-1/2}(\mathbf{A}\mathbf{K}_d^{-1/2})^+\mathbf{A} \rfloor \; \mathbf{K}_d^{-1}\mathbf{K}_o \tag{38a}$$

$$\mathbf{F}^*=\lfloor \mathbf{I}-\mathbf{K}_d^{-1/2}(\mathbf{A}\mathbf{K}_d^{-1/2})^+\mathbf{A} \rfloor \; \mathbf{K}_d^{-1}\mathbf{F} \tag{38b}$$

where

$$\mathbf{K}_d=\begin{bmatrix}\mathbf{k}^1_{lsls} & 0 & 0 & 0\\ 0 & \mathbf{k}^1_{bb} & 0 & 0\\ 0 & 0 & \mathbf{k}^2_{bb} & 0\\ 0 & 0 & 0 & \mathbf{k}^2_{ll}\end{bmatrix},\ \mathbf{K}_o=\begin{bmatrix}0 & \mathbf{k}^1_{lsb} & 0 & 0\\ \mathbf{k}^1_{bls} & 0 & 0 & 0\\ 0 & 0 & 0 & \mathbf{k}^2_{bl}\\ 0 & 0 & \mathbf{k}^2_{lb} & 0\end{bmatrix}$$

$$\mathbf{F}=\begin{bmatrix}\mathbf{f}^1_{ls}\\ \mathbf{f}^1_{b}\\ 0\\ \mathbf{f}^2_{l}\end{bmatrix}-\begin{bmatrix}\mathbf{k}^1_{lslm}\mathbf{u}^1_{lm}\\ \mathbf{k}^1_{blm}\mathbf{u}^1_{lm}\\ 0\\ 0\end{bmatrix}$$

Also, $\mathbf{A}$ is Boolean matrix. The solution of the equilibrium equation with respect to $\mathbf{u}_c$ is obtained as

$$\mathbf{u}_c=(\mathbf{K}^*)^{-1}\mathbf{Q}(\mathbf{F}_A-\mathbf{K}_B\mathbf{u}^1_{lm}) \tag{39}$$

where

$$\mathbf{Q}=\lfloor \mathbf{I}-\mathbf{K}_d^{-1/2}(\mathbf{A}\mathbf{K}_d^{-1/2})^{+}\mathbf{A}\rfloor\ \mathbf{K}_d^{-1}$$

$$\mathbf{F}_A=[\mathbf{f}^1_{ls}\ \mathbf{f}^1_{b}\ 0\ \mathbf{f}^2_{l}]^T$$

$$\mathbf{K}_B=[\mathbf{k}^1_{lslm}\ \mathbf{k}^1_{blm}\ 0\ 0]^T$$

Substituting equation (39) into equation (37a) and arranging the result, the displacement vector $\mathbf{u}^1_{lm}$ can be obtained in terms of

$$\mathbf{u}^1_{lm}=\mathbf{R}^{-1}\mathbf{P} \tag{40}$$

where

$$\mathbf{R}=\mathbf{k}^1_{lmlm}-\mathbf{K}_A(\mathbf{K}^*)^{-1}\mathbf{Q}\mathbf{K}_B$$

$$\mathbf{K}_A=\lfloor \mathbf{k}^1_{lmls}\ \mathbf{k}^1_{lmb}\ 0\ 0\rfloor$$

$$\mathbf{P}=\mathbf{f}^1_{lm}-\mathbf{K}_A(\mathbf{K}^*)^{-1}\mathbf{Q}\mathbf{F}_A$$

Equation (40) represents the displacement vector of master modes to eliminate the slave mode displacements. The final equation exhibits that the static behavior of an entire structure subjected to linear constraints can be explained by only master modes from the proposed method.

6. Application II

The proposed method can be effectively applied to the static analysis of the structure of disturbed stress distribution and irregular geometry. As a simple application, consider a beam with a rectangular opening subjected to concentrated loads as shown by Fig. 4(a). The beam is fixed at an end and is simply supported at the other end.

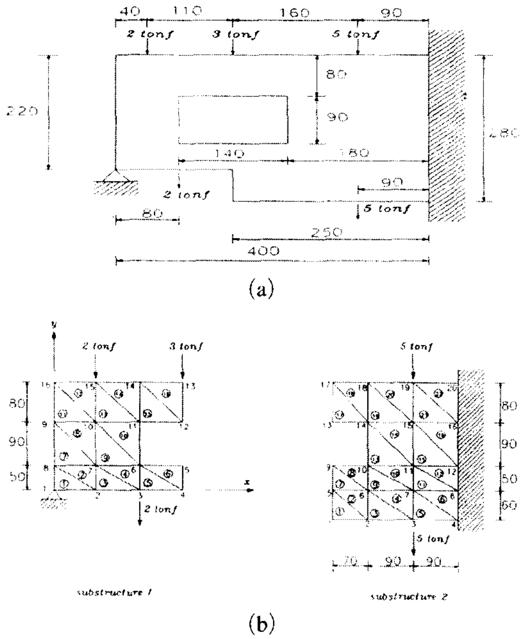


Fig. 4 A beam fixed at an end and simply supported at the other end (unit: mm); (a) An entire structure, (b) Two subdivided structures

To perform the static analysis of the entire structure, the proposed method can be utilized by partitioning into two substructures and synthesizing them as shown by Fig. 4(b). Based on the finite element process of plane stress, the equilibrium equation of each substructure of Fig. 4(b) can be written as

$$\mathbf{K}^{(1)}\mathbf{u}^{(1)}=\mathbf{F}^{(1)} \tag{41a}$$

$$\mathbf{K}^{(2)}\mathbf{u}^{(2)}=\mathbf{F}^{(2)} \tag{41b}$$

where the superscripts (1) and (2) denote the substructures 1 and 2, respectively.

As a result of partitioning into two substructures, the support condition at the interface takes free end. Thus, the substructure 1 is an unstable structure which can not independently determine the nodal displacements. The nodal displacements of the entire structure can be calculated by synthesizing the equilibrium equations of two substructures and the compatibility conditions at the interface. Applying equations (38), the static behavior of the entire structure can be explicitly determined. The following data for numerical output were utilized :

$$E = 2 \times 10^5 \text{ kgf/cm}^2, \; v = 0.3$$

The nodal displacements obtained from the proposed method represent the same results as the finite element analysis of the entire structure not to be partitioned. From this application, it can be verified that the proposed method is applied to the complicated structures of geometrical irregularity and disturbed stress state.

7. Conclusions

This study provided the equilibrium equation of constrained structures composed of fixed-free and/or free-free substructures. It was shown that the equilibrium equation of constrained structures and the constraint forces can be obtained by minimizing the total potential energy with respect to the displacement vector and the constraint force vector, respectively. The provided method gives the explicit solutions without depending on any numerical analysis and can be applied to the structural analysis of free-free and fixed-free substructures with linear constraints. Although the proposed method requires the repeated calculation at interface nodes of substructures, this difficulty will be overcome by reducing the number of degrees of freedom of interior substructures through the static condensation approach. Applications illustrated the effectiveness and easiness of the proposed method. It was shown that the static behavior of an entire structure subjected to linear constraints can be explained by only selected master modes.

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